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Shift-invariant spaces of tempered distributions and L_p-functions

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Abstract

This paper studies the structure of shift-invariant spaces. A characterization for the univariate shift-invariant spaces of tempered distributions is given. In L_p case, an inclusive relation in terms of Fourier transform is established.

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1. Introduction

This paper studies the structure of shift-invariant spaces generated by finitely many compactly supported distributions.

A linear space S of distributions is *shift-invariant* if, for any $\phi \in S$ and any $\alpha \in \mathbb{Z}^s$, $\phi(\cdot - \alpha) \in S$.

The space $\mathscr{S}(\mathbb{R}^s)$ of rapidly decreasing functions is the set of all infinitely differentiable functions t such that, for any polynomial $p(x)$ and $\alpha \in \mathbb{Z}^s_+$,

$$|p(x)\partial^\alpha t(x)| \to 0, \quad |x| \to \infty.$$

The dual space $\mathscr{S}'(\mathbb{R}^s)$ is the space of tempered distributions. Similarly, $\mathscr{S}(\mathbb{Z}^s) \subseteq \ell(\mathbb{Z}^s)$ consists of all sequences that decay faster than the reciprocal of any

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polynomial sequence, where $\ell(\mathbb{Z}^s)$ is the set of all complex valued sequences defined on $\mathbb{Z}^s$. A sequence b is in $\mathscr{S}'(\mathbb{Z}^s)$, the dual space of $\mathscr{S}(\mathbb{Z}^s)$, if and only if b is of polynomial growth, i.e., there exists a polynomial p satisfying $|b(\alpha)| \leqslant |p(\alpha)|, \alpha \in \mathbb{Z}^s$.

For a distribution ϕ and a sequence $b \in \ell(\mathbb{Z}^s)$, the *semi-convolution* $\phi *' b$ is defined by

$$\phi *' b = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha)\phi(\cdot - \alpha).$$

It is well defined in case either ϕ is compactly supported or $b \in \ell_0(\mathbb{Z}^s)$, where $\ell_0(\mathbb{Z}^s)$ is the subspace of finitely supported sequences in $\ell(\mathbb{Z}^s)$. In both cases, it is clear that $\phi *' b \in \mathscr{D}'(\mathbb{R}^s)$, the space of distributions on $C_0^\infty(\mathbb{R}^s)$ of compactly supported and infinitely differentiable functions.

Throughout this paper we assume that Φ is a finite set consisting of $\phi_j \in \mathscr{S}'(\mathbb{R}^s)$, $j = 1, \dots, m$. We define the linear space $S_0(\Phi)$ as the smallest shift-invariant space containing Φ, that is,

$$S_0(\Phi) = \left\{ \sum_{1 \leqslant j \leqslant m} \phi_j *' b_j \colon\ b_j \in \ell_0(\mathbb{Z}^s),\ 1 \leqslant j \leqslant m \right\}.$$

For $1 \leqslant p \leqslant \infty$, by $L_p(\mathbb{R}^s)$ we denote the Banach space of all complex valued measurable functions on $\mathbb{R}^s$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$. The Fourier transform $\hat{f}$, for $f \in L_1(\mathbb{R}^s)$, is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}^s} f(x) e^{-ix\omega} \, d\omega, \quad \omega \in \mathbb{R}^s.$$

The Fourier transform of a distribution in $\mathscr{S}'(\mathbb{R}^s)$ is defined by duality. It is well known that the Fourier transform of a compactly supported distribution can be extended to an entire function of $z \in \mathbb{C}^s$.

For compactly supported distributions $\phi_j \in \mathscr{S}'(\mathbb{R}^s)$, $j = 1, \dots, m$, the shift-invariant space $S(\Phi) \subseteq \mathscr{D}'(\mathbb{R}^s)$ is given by

$$S(\Phi) = \left\{ \sum_{1 \leqslant j \leqslant m} \phi_j *' b_j \colon\ b_j \in \ell(\mathbb{Z}^s),\ 1 \leqslant j \leqslant m \right\}.$$

Further, if Φ consists of finitely many functions in $L_p(\mathbb{R}^s)$, denote by $S_p(\Phi)$ the closure of $S_0(\Phi)$ in $L_p(\mathbb{R}^s)$. It is the smallest closed shift-invariant space in $L_p(\mathbb{R}^s)$ that contains Φ.

It is a meaningful problem, for a shift-invariant subspace $S \subseteq S(\Phi)$, to characterize the elements of S. The following result on this direction is due to de Boor et al. [2], which holds for any finite set $\Phi \subseteq L_2(\mathbb{R}^s)$.

$$S_2(\Phi) = \left\{ f \in L_2(\mathbb{R}^s)\colon \hat{f} = \sum_{1 \leqslant j \leqslant m} \hat{\phi}_j a_j, a_j \text{ is } 2\pi\mathbb{Z}^s\text{-periodic}, \ 1 \leqslant j \leqslant m \right\}. \tag{1.1}$$

The characterization (1.1) plays an important role in wavelet analysis as well as the approximation order by $S_2(\Phi)$, see [1–4] etc. In [6], Jia gave another proof of (1.1). It was proved in [6] that, for $\Phi \subseteq L_p(\mathbb{R}^s), S(\Phi) \cap L_p(\mathbb{R}^s)$ is closed in $L_p(\mathbb{R}^s)$ and that $S(\Phi) \cap L_2(\mathbb{R}^s) = S_2(\Phi)$ for $\Phi \subseteq L_2(\mathbb{R}^s)$.

It is well known that, for any $p \in [1,2]$ and any function $f \in L_p(\mathbb{R}^s)$, the Fourier transform $\hat{f} \in L_{p'}(\mathbb{R}^s)$, where p' is the conjugate number of $p, 1/p + 1/p' = 1$. The analogue of the subspace in the right-hand side of (1.1) is defined as follows:

$$X_p(\Phi) = \left\{ f \in L_p(\mathbb{R}^s)\colon \hat{f} = \sum_{1 \leqslant j \leqslant m} \hat{\phi}_j a_j, a_j \text{ is } 2\pi\mathbb{Z}^s\text{-periodic}, \ 1 \leqslant j \leqslant m \right\}. \tag{1.2}$$

In this paper, we are interested in the shift-invariant spaces in $\mathscr{S}'(\mathbb{R}^s)$ and $L_p(\mathbb{R}^s)$, respectively. When Φ consists of compactly supported distributions in $\mathscr{S}'(\mathbb{R}^s)$, we give a characterization in Section 2 for $f \in \mathscr{S}'(\mathbb{R}^s)$ by the coefficients $b_j(\alpha), \ \alpha \in \mathbb{Z}^s, 1 \leqslant j \leqslant m$, provided that either $s = 1$ or the shifts of Φ are stable (see below for the definition of stable). For $\Phi \subseteq L_p(\mathbb{R}^s), \ 1 \leqslant p \leqslant 2$, consisting of compactly supported functions, we establish in Section 3 that $S(\Phi) \cap L_p(\mathbb{R}^s) \subseteq X_p(\Phi)$. As a by product we present a proof of the equality $S(\Phi) \cap L_2(\mathbb{R}^s) = S_2(\Phi)$ for $\Phi \subseteq L_2(\mathbb{R}^s)$.

There are two methods to describe the structure of shift-invariant spaces. Fourier transforms of $\phi_j, 1 \leqslant j \leqslant m$, have been used extensively and mainly to characterize the shift-invariant subspaces in $L_2(\mathbb{R}^s)$. Another method is to use semi-convolution. It turns out that the semi-convolution is a powerful tool to deal with the non-L_2 case as well as L_2 case. Our method is a combination of them.

2. Shift-invariant subspaces of tempered distributions

In this section we characterize the shift-invariant subspaces of $\mathscr{S}'(\mathbb{R}^s)$ in terms of coefficients in semi-convolution.

We first introduce some notions and results about the linear independence and stability of the shifts of Φ. Let $\Phi = \{\phi_j\}_{1 \leqslant j \leqslant m}$ be a finite set consisting of compactly supported distributions. The shifts of Φ are *linearly independent* (*stable*, respectively) if for any $b_j \in \ell(\mathbb{Z}^s)(\ell_\infty(\mathbb{Z}^s)$, respectively), $1 \leqslant j \leqslant m$,

$$\sum_{1 \leqslant j \leqslant m} \sum_{\alpha \in \mathbb{Z}^s} b_j(\alpha)\phi_j(\cdot - \alpha) = 0 \Rightarrow b_j(\alpha) = 0, \quad 1 \leqslant j \leqslant m, \ \alpha \in \mathbb{Z}^s. \tag{2.1}$$

Let Φ consist of finitely many compactly supported distributions $\phi_j \in \mathscr{S}'(\mathbb{R}^s)$, $1 \leqslant j \leqslant m$. The restriction of $S(\Phi)$ to the cube $\Omega = (-1,1)^s$ is a finite dimensional subspace of $\mathscr{S}'(\mathbb{R}^s)$. There exists a basis $\psi_k, \ 1 \leqslant k \leqslant n$, of $S(\Phi)|_\Omega$. For any $b_k \in \ell(\mathbb{Z}^s)$, $\sum_{1 \leqslant k \leqslant n} \psi_k *' b_k = 0$ if and only if $b_k = 0, \ 1 \leqslant k \leqslant n$.

Obviously, there exists $m \times n$ finitely supported sequences $c_{jk},\ 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n$, such that for any $\alpha \in \mathbb{Z}^s$

$$\phi_j(\cdot + \alpha)|_\Omega = \sum_{1 \leqslant k \leqslant n} c_{jk}(\alpha)\psi_k. \tag{2.2}$$

Define the Laurent polynomials as

$$g_{jk}(z) = \sum_{\alpha \in \mathbb{Z}^s} c_{jk}(\alpha) z^{-\alpha}, \quad z \in (\mathbb{C}\backslash\{0\})^s,\ 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n, \tag{2.3}$$

and the matrix

$$G(z) = (g_{jk}(z))_{1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n}, \quad z \in (\mathbb{C}\backslash\{0\})^s. \tag{2.4}$$

Recall from [7] that the linear independence (stability, respectively) of the shifts of Φ is equivalent to the fact that, for any $z \in (\mathbb{C}\backslash\{0\})^s (z \in \mathbb{T}^s$, respectively) the matrix $G(z)$ has rank m.

The difference operator is a convenient tool. Given $\alpha \in \mathbb{Z}^s$, the operator τ^α on $\ell(\mathbb{Z}^s)$ is defined by $\tau^\alpha f = f(\cdot + \alpha),\ f \in \ell(\mathbb{Z}^s)$. A Laurent polynomial $p(z) = \sum_\alpha c(\alpha) z^\alpha$ induces a difference operator $p(\tau) = \sum_\alpha c(\alpha)\tau^\alpha$.

Denote by V the set of all the distributions that have the following representation: there exist some $f_k \in \ell(\mathbb{Z}^s),\ 1 \leqslant k \leqslant n$, such that for all $\alpha \in \mathbb{Z}^s$,

$$f(\cdot + \alpha)|_\Omega = \sum_{1 \leqslant k \leqslant n} f_k(\alpha)\psi_k. \tag{2.5}$$

Clearly, $S(\Phi) \subseteq V$. Moreover, a distribution $f \in V$ with representation (2.5) belongs to $S(\Phi)$ if and only if the system

$$\sum_{1 \leqslant j \leqslant m} g_{jk}(\tau) b_j = f_k, \quad 1 \leqslant k \leqslant n \tag{2.6}$$

is solvable for $b_j,\ 1 \leqslant j \leqslant m$. The details are referred to, for example, [7].

Based on Toeplitz Theorem, Jia [5] characterized the solvability of (2.6) as follows.

Lemma 1 (Jia [5]). *Assume that for any pair of (j, β), there are only finitely many pairs of (k, α) such that $c_{jk}(\alpha - \beta) \neq 0$. Then the system of difference equations* (2.6) *is solvable if and only if it satisfies that, for any Laurent polynomials $q_k(z),\ 1 \leqslant k \leqslant n$,*

$$\sum_{1 \leqslant k \leqslant n} g_{jk} q_k = 0 \quad \forall 1 \leqslant j \leqslant m \Rightarrow \sum_{1 \leqslant k \leqslant n} q_k(\tau) f_k = 0. \tag{2.7}$$

Our first result is a characterization of $f \in V$ to be a distribution in $\mathscr{S}'(\mathbb{R}^s)$ in terms of its coefficient sequences $f_k,\ 1 \leqslant k \leqslant n$. Its proof is elementary and, therefore, omitted.

Lemma 2. *Let $f \in V$ be as given in* (2.6) *with the coefficients $f_k(\alpha),\ 1 \leqslant k \leqslant n,\ \alpha \in \mathbb{Z}^s$. Then $f \in \mathscr{S}'(\mathbb{R}^s)$ if and only if for any $k,\ 1 \leqslant k \leqslant n,\ f_k \in \mathscr{S}'(\mathbb{Z}^s)$.*

The following lemma is a well-known result.

Lemma 3. *For any nonzero univariate Laurent polynomial* $q(z)$ *and a sequence* $g \in \mathscr{S}'(\mathbb{Z})$, *the difference equation* $q(\tau)a = g$ *has solution* $a \in \mathscr{S}'(\mathbb{Z})$. *The solution is unique if and only if* $q(z)$ *has no zero on* $\mathbb{T}$.

Theorem 4. *Assume that* $\Phi = \{\phi_j\}_{1 \leqslant j \leqslant m}$ *is a finite set of compactly supported distributions in* $\mathscr{S}'(\mathbb{R}^s)$. *For univariate case*, $s = 1$, *we have*

$$S(\Phi) \cap \mathscr{S}'(\mathbb{R}^s) = \left\{ f \colon f = \sum_{1 \leqslant j \leqslant m} \phi_j *' b_j, b_j \in \mathscr{S}'(\mathbb{Z}^s) 1 \leqslant j \leqslant m \right\}. \tag{2.8}$$

Proof. It is clear that for any $b_j \in \mathscr{S}'(\mathbb{Z}^s), \sum_{1 \leqslant j \leqslant m} \phi_j *' b_j \in S(\Phi) \cap \mathscr{S}'(\mathbb{R}^s)$. Assume now $f \in S(\Phi) \cap \mathscr{S}'(\mathbb{R}^s)$. Then $f \in V$. There exist coefficient sequences $f_k \in \mathscr{S}'(\mathbb{Z}), 1 \leqslant k \leqslant n$, such that (2.5) holds. As we have known that, for such f_k, $1 \leqslant k \leqslant n$, the system of difference equations (2.6) is solvable for some b_j, $1 \leqslant j \leqslant m$. Therefore the implication relation (2.7) holds.

We shall appeal to the results for the decompositions of (univariate) polynomial matrices to establish the solvability of (2.6) for $b_j \in \mathscr{S}'(\mathbb{Z})$, $1 \leqslant j \leqslant m$. Recall that $G(z)$ is the Laurent polynomial matrix in (2.3). It is well known that there exist an $n \times n$ Laurent polynomial matrix $L(z)$ and a $m \times m$ Laurent polynomial matrix $R(z)$ satisfying $\det L(z) = \det R(z) = 1, z \in \mathbb{C}$, and $L(z)G^T(z) = D^T(z)R(z), z \in \mathbb{C}$, where

$$D(z) = (d_{jk}(z))_{1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n}$$

is a Laurent polynomial matrix with the only nonzero entries $d_{jj}(z)$, $j = 1, 2, \ldots, r$, and r being the largest number such that there exists $r \times r$ submatrix of $G(z)$ whose determinant is not identically zero. Moreover, $d_{jj}(z)$ is a factor of $d_{j+1j+1}(z)$, $j = 1, \ldots, r-1$.

As usual, the operator $L(\tau)$ on $(\ell(\mathbb{Z}))^n$ is induced by matrix $L(z)$. Moreover, as Lemma 3, we may conclude that $L(\tau)$ is an invertible operator on $(\mathscr{S}'(\mathbb{Z}))^n$. Similarly $R(\tau)$ is invertible on $(\mathscr{S}'(\mathbb{Z}))^m$. Let $b = (b_1, \ldots, b_m)^T$ and $f = (f_1, \ldots, f_n)^T$. Then we may rewrite (2.6) as $G^T(\tau)b = f$. Therefore, it is obvious that (2.6) has solution $b_j \in \mathscr{S}'(\mathbb{Z}), 1 \leqslant j \leqslant m$, if and only if the system of difference equations

$$\sum_{1 \leqslant j \leqslant m} d_{jk}(\tau) a_j = h_k, \quad 1 \leqslant k \leqslant n, \tag{2.9}$$

has solution $a_j \in \mathscr{S}'(\mathbb{Z})$, $1 \leqslant j \leqslant m$, where h_k, $1 \leqslant k \leqslant n$, are given by

$$h = (h_1, \ldots, h_n)^T = L(\tau)f \in (\mathscr{S}'(\mathbb{Z}))^n.$$

We first conclude that $h_i = 0$ for $i = r+1, \ldots, n$. To this end denote

$$L(z) = (l_{ik}(z))_{1 \leqslant i,k \leqslant n}.$$

Recall that all entries in the last $n - r$ rows of $D^T(z)$ are zero. So does $L(z)G^T(z)$.

This means

$$\sum_{1\leqslant k\leqslant n} g_{jk}l_{ik}=0 \quad \forall 1\leqslant j\leqslant m,\ \ i=r+1,\dots,n.$$

It follows from (2.7) that

$$h_i=\sum_{1\leqslant k\leqslant n} l_{ik}(\tau)f_k=0, \quad i=r+1,\dots,n,$$

as claimed.

By Lemma 3, for any $j, 1\leqslant j\leqslant r$, there exists $a_j\in\mathscr{S}'(\mathbb{Z})$ satisfying $d_{jj}(\tau)a_j=h_j$. Setting $a_j=0,\ j=r+1,\dots,m,$ and $b=(b_1,\dots,b_m)^T=(R(\tau))^{-1}a$, then $b_j\in\mathscr{S}'(\mathbb{Z}),\ 1\leqslant j\leqslant m$, satisfies (2.6). The proof is complete. □

To establish equality (2.8) in multivariate case we need the stability of the shifts of Φ.

Theorem 5. *Assume that $\Phi=\{\phi_j\}_{1\leqslant j\leqslant m}$ is a finite set of compactly supported distributions in $\mathscr{S}'(\mathbb{R}^s)$ with stable shifts. Then* (2.8) *holds for any s.*

Proof. Assume that $f\in S(\Phi)\cap\mathscr{S}'(\mathbb{R}^s)$. There are n coefficient sequences $f_k\in\ell(\mathbb{Z}^s), 1\leqslant k\leqslant n$, such that (2.5) holds. By Lemma 2, $f_k\in\mathscr{S}'(\mathbb{Z}^s),\ 1\leqslant k\leqslant n$. Since the shifts of Φ are stable, the matrix $G(z)$ given as in (2.4) has rank m for any $z\in\mathbb{T}^s$. The system of difference equations (2.6) satisfies (2.7) for that $f_k,\ 1\leqslant k\leqslant n$. Similar to the proof of [6, Theorem 7.1] we can deduce that the system of difference equations (2.6) is uniquely solvable for $b_j\in\mathscr{S}'(\mathbb{Z}^s),\ 1\leqslant j\leqslant m$. We refer to [6] for the details. The proof is complete. □

3. Shift-invariant spaces in $L_p(\mathbb{R}^s)(1\leqslant p\leqslant 2)$

We establish in this section the inclusive relation $S(\Phi)\cap L_p(\mathbb{R}^s)\subseteq X_p(\Phi)$ for $\Phi\in L_p(\mathbb{R}^s),\ 1\leqslant p\leqslant 2$.

Assume throughout this section that the finite set $\Phi\subseteq L_p(\mathbb{R}^s),\ 1\leqslant p\leqslant 2$, consists of compactly supported functions. We may consider the restriction of $S(\Phi)$ to the cube $[0,1]^s$, instead of $(-1,1)^s$ as in Section 2. Thus, in contrast to Section 2, the basis $\psi_k,\ 1\leqslant k\leqslant n$, of $S(\Phi)|_{[0,1]^s}$ can be so chosen that $\psi_k,\ 1\leqslant k\leqslant n$, are supported on $[0,1]^s$. As a finite set $\Psi:=\{\psi_k\}_{1\leqslant k\leqslant n}$, it satisfies that, for any numbers $c_k\in\mathbb{C},\ 1\leqslant k\leqslant n$,

$$\left\|\sum_{1\leqslant k\leqslant n} c_k\psi_k\right\|_{L_p[0,1]^s}\approxeq\left(\sum_{1\leqslant k\leqslant n}|c_k|^p\right)^{1/p},$$

from which it follows that, for any $f_k \in \ell_p(\mathbb{Z}^s), 1 \leqslant k \leqslant n$,

$$\left\| \sum_{1 \leqslant k \leqslant n} \psi_k *' f_k \right\|_p \approxeq \left(\sum_{1 \leqslant k \leqslant n} \|f_k\|_p \right)^{1/p}. \tag{3.1}$$

It is always true that $S(\Phi) \subseteq S(\Psi)$. Instead of (2.2), it holds that for some $m \times n$ finitely supported sequences $c_{jk}, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n$,

$$\phi_j = \sum_{1 \leqslant k \leqslant n} \psi_k *' c_{jk}, \quad 1 \leqslant j \leqslant m. \tag{3.2}$$

Taking Fourier transform, we get

$$\hat{\phi}_j(\omega) = \sum_{1 \leqslant k \leqslant n} \hat{\psi}_k(\omega) g_{jk}(e^{i\omega}), \quad 1 \leqslant j \leqslant m, \tag{3.3}$$

whereas in Section 2, $g_{jk}(z) = \sum_{\alpha \in \mathbb{Z}^s} c_{jk}(\alpha) z^{-\alpha}, \ 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant n.$

As in Section 2, an element $f = \sum_{1 \leqslant k \leqslant n} \psi_k *' f_k \in S(\Phi)$ if and only if the system of difference equations (2.6) is solvable or, equivalently, the implication relation (2.7) holds.

Theorem 6. *Let Φ be a finite set of compactly supported functions in $L_p(\mathbb{R}^s)$. Then*

$$S(\Phi) \cap L_p(\mathbb{R}^s) \subseteq X_p(\Phi), \quad p \in [1, 2],$$

where $X_p(\Phi)$ is as defined in (1.2).

Proof. Let $f \in S(\Phi) \cap L_p(\mathbb{R}^s)$. Then $f = \sum_{1 \leqslant k \leqslant n} \psi_k *' f_k$ with $f_k \in l_p(\mathbb{Z}^s), \ 1 \leqslant k \leqslant n$, by (3.1). By Hausdorff–Young Theorem, for any $1 \leqslant k \leqslant n$, the series $\sum_{\alpha \in \mathbb{Z}^s} f_k(\alpha) e^{i\alpha\omega}$ converges in $L_{p'}[0, 2\pi]^s$ to a function, say, $\tilde{f}_k(\omega)$, where p' is the conjugate number of $p, \ 1/p + 1/p' = 1$.

On the other hand, since system (2.6) is solvable for $b_j, \ 1 \leqslant j \leqslant m$, it satisfies (2.7) by Lemma 1. Therefore, for any Laurent polynomials $q_k, \ 1 \leqslant k \leqslant n$,

$$\sum_{1 \leqslant k \leqslant n} q_k g_{jk} = 0 \ \forall 1 \leqslant j \leqslant m \Rightarrow \sum_{1 \leqslant k \leqslant n} q_k(e^{-i\omega}) \tilde{f}_k(\omega) = 0 \quad \text{a.e. } \omega. \tag{3.4}$$

We shall prove that, for a.e. $\omega \in [0, 2\pi]^s$, the following system is solvable.

$$\sum_{1 \leqslant j \leqslant m} g_{jk}(e^{i\omega}) a_j = \tilde{f}_k(\omega), \quad 1 \leqslant k \leqslant n. \tag{3.5}$$

Let N be the largest integer such that there exists a submatrix $M(z)$ with size N of $G(z)$ satisfying $\det M(z)$ is not identically zero. As a Laurent polynomial $\det M(z) \neq 0$ for a.e. z. Without loss of any generality, we assume

$$M(z) = (g_{jk}(z))_{1 \leqslant j,k \leqslant N}.$$

Given any $2\pi\mathbb{Z}^s$-periodic functions $a_j(\omega), N+1 \leqslant j \leqslant m$, the system

$$\sum_{1 \leqslant j \leqslant N} g_{jk}(e^{i\omega}) a_j = \tilde{f}_k(\omega) - \sum_{N+1 \leqslant j \leqslant m} g_{jk}(e^{i\omega}) a_j(\omega), \quad 1 \leqslant k \leqslant N \tag{3.6}$$

has unique solution $a_j := a_j(\omega),\ 1\leqslant j\leqslant N$, for a.e. ω. Clearly, $a_j(\omega)$ are also $2\pi\mathbb{Z}^s$-periodic functions. Therefore, $a_j(\omega),\ 1\leqslant j\leqslant m$, solve the first N equations in (3.5).

It needs to verify that $a_j(\omega),\ 1\leqslant j\leqslant m$, satisfy the last $n-N$ equations in (3.5). In fact, for any $k_0,\ N<k_0\leqslant n$, by the definitions of N and $M(z)$, there exist Laurent polynomials $q_k,\ 1\leqslant k\leqslant N$, and q_{k_0}, not all zero, such that

$$q_{k_0}(z)g_{jk_0}(z)+\sum_{1\leqslant k\leqslant N} q_k(z)g_{jk}(z)=0,\quad z\in\mathbb{T}^s,\ 1\leqslant j\leqslant m. \tag{3.7}$$

It is clear that q_{k_0} is not identically zero, for otherwise, the above equalities imply $\det M(z)=0,\ z\in\mathbb{T}^s$, a contradiction. It follows from Lemma 1 and (3.4) that

$$\tilde{f}_{k_0}(\omega)=-\sum_{1\leqslant k\leqslant N} q_k(e^{-i\omega})\tilde{f}_k(\omega)/q_{k_0}(e^{-i\omega})\quad \text{a.e. } \omega. \tag{3.8}$$

By (3.7), (3.6) and (3.8), we obtain that for a.e. ω

$$\begin{aligned}
&\sum_{1\leqslant j\leqslant m} g_{jk_0}(e^{i\omega})a_j(\omega)\\
&\quad=-\sum_{1\leqslant k\leqslant N}\ \sum_{1\leqslant j\leqslant m} g_{jk}(e^{i\omega})a_j(\omega)q_k(e^{-i\omega})/q_{k_0}(e^{-i\omega})\\
&\quad=-\sum_{1\leqslant k\leqslant N} q_k(e^{-i\omega})\tilde{f}_k(\omega)/q_{k_0}(e^{-i\omega})=\tilde{f}_{k_0}(\omega),
\end{aligned}$$

as desired. It follows from equalities (3.3) and (3.5) that

$$\hat{f}=\sum_{1\leqslant j\leqslant m}\hat{\phi}_j a_j\quad \text{a.e. } \omega. \tag{3.9}$$

The proof is complete. □

Corollary 7 (Jia [6, Theorem 4.1]). *Let Φ be a finite set of compactly supported functions in $L_2(\mathbb{R}^s)$. Then*

$$S(\Phi)\cap L_2(\mathbb{R}^s)=S_2(\Phi).$$

Proof. Recall that $S_2(\Phi)$ is the smallest closed shift-invariant space containing Φ. Since $S(\Phi)\cap L_2(\mathbb{R}^s)$ is a closed subspace of $L_2(\mathbb{R}^s)$ containing Φ [6], $S_2(\Phi)\subseteq S(\Phi)\cap L_2(\mathbb{R}^s)$. By Theorem 6, it suffices to establish $X_2(\Phi)\subseteq S_2(\Phi)$.

Suppose that $g\in S_2(\Phi)^{\perp}$, the orthogonal complement of $S_2(\Phi)$ in $L_2(\mathbb{R}^s)$. Then

$$\int_{\mathbb{R}^s}\phi_j(x-\alpha)\overline{g(x)}\,dx=0,\quad 1\leqslant j\leqslant m,\ \alpha\in\mathbb{Z}^s.$$

Therefore, by Parseval identity, for any $j, 1\leqslant j\leqslant m$,

$$\sum_{\alpha\in\mathbb{Z}^s}\hat{\phi}_j(\omega+2\pi\alpha)\overline{\hat{g}(\omega+2\pi\alpha)}=0\quad \text{a.e. } \omega.$$

For any $f \in X_2(\Phi)$, there are some $2\pi\mathbb{Z}^s$-periodic functions $a_j,\ 1 \leqslant j \leqslant m$, such that (3.9) holds. By the above equalities

$$\sum_{\alpha \in \mathbb{Z}^s} \hat{f}(\omega + 2\pi\alpha)\overline{\hat{g}(\omega + 2\pi\alpha)} = 0 \quad \text{a.e. } \omega.$$

This in turn gives $\int_{\mathbb{R}^s} f(x)\overline{g(x)}\,dx = 0$. Therefore, $g \in X_2(\Phi)^{\perp}$. It follows that $S_2(\Phi)^{\perp} \subseteq X_2(\Phi)^{\perp}$. The proof is complete. □

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